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Remarks on the equivalence between the shape-invariance condition and the factorisation condition

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Abstract. It is shown that the shape-invariance condition for supersymmetric potentials and the factorisation condition for Sturm-Liouville eigenvalue problems are equivalent. The roots of this self-consistency condition in supersymmetric quantum mechanics and the factorisation method are found in the theory of Riccati equations.

It is well known [1] that supersymmetric quantum mechanics [2] (SUSYQM)—the realisation of the algebra

$$Q^2 = (Q^*)^2 = 0 \tag{1a}$$

$$\{Q, Q^*\} = H \tag{1b}$$

$$[H, Q] = [H, Q^*] = 0 \tag{1c}$$

in terms of linear differential operators of first order—is identical to the factorisation method [3]. Thus the algebra (1) appears as the natural structure underlying the far-reaching concept [4, 5] of solving Sturm-Liouville (Schrödinger) eigenvalue problems via a factorisation of the differential equation into a product of differential operators of first order.

It is the purpose of this paper to show that the factorisation condition (a criterion for the applicability of the factorisation method to an eigenvalue problem) is identical to the shape-invariance condition [6]. (The shape-invariance condition (discussed below, equation (9)) establishes the supersymmetry of a pair of associated Hamiltonians in terms of the potentials and allows an elegant algebraic solution of a given Schrödinger equation (see Khare and Sukhatme [1], also [6, 7]).) Since both conditions are, in fact, based on the equivalence between a linear differential equation of second order and an associated Riccati equation [8], the extension of SUSYQM (and the factorisation method) to scattering problems (continuous eigenvalues) [1, 7] can easily be done [5].

The realisation of the algebra (1) in the form

$$Q \equiv \frac{1}{2^{3/2}} (p - \mathrm{i} W(x))(\sigma_1 + \mathrm{i}\sigma_2) \tag{2a}$$

leads to the supersymmetric Hamiltonian ($\hbar \equiv 1$; σ_i are Pauli spin matrices)

$$H = \tfrac{1}{2}(p^2 + W^2(x) + \sigma_3 W'(x)) \tag{2b}$$

with the superpotential $W(x)$.

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The spectrum of the Hamiltonian in (2*b*) is non-negative and the levels are, in general, degenerate. If the supersymmetry is not spontaneously broken, the ground state is non-degenerate and the ground-state energy is exactly zero; the corresponding wavefunction is normalisable [1, 9].

Thus the 'Hamiltonian' of SUSYQM is the pair of Hamiltonians, $H_\pm$, generated by the construction above:

$$\begin{aligned} H_\pm &= \tfrac{1}{2}p^2 + (W^2(x) \pm W'(x))/2 \\ &\equiv \tfrac{1}{2}p^2 + V_\pm(x). \end{aligned} \tag{3}$$

The spectrum of these Hamiltonians is identical (apart from the eigenvalue zero) and their (normalised) eigenfunctions are paired. This can be shown using the definitions

$$Q \equiv \begin{pmatrix} 0 & A_+ \\ 0 & 0 \end{pmatrix} \qquad Q^* \equiv \begin{pmatrix} 0 & 0 \\ A_- & 0 \end{pmatrix} \tag{4a}$$

where

$$A_\pm \equiv \frac{1}{\sqrt{2}}(p \mp \mathrm{i}W(x)). \tag{4b}$$

The Schrödinger equation ($E \neq 0$)

$$H_+\psi = E\psi = A_+A_-\psi \tag{5a}$$

then implies

$$H_-(A_-\psi) = E(A_-\psi) \tag{5b}$$

and

$$H_-\varphi = E'\varphi = A_-A_+\varphi \tag{5c}$$

implies

$$H_+(A_+\varphi) = E'(A_+\varphi) \tag{5d}$$

with multiplicities preserved. The eigenfunctions are connected by

$$\varphi = E^{-1/2}A_-\psi \qquad \psi = E^{-1/2}A_+\varphi. \tag{6}$$

(We use in the following the convention that the wavefunction ψ_0 of the ground state of H_-, given by

$$\psi_0(x) \sim \exp\left(-\int W(x')\,\mathrm{d}x'\right) \tag{7}$$

is assumed to be normalisable.)

If now the potentials

$$V_\pm(x) \equiv \frac{W^2(x) \pm W'(x)}{2} \tag{8}$$

of the partner Hamiltonians $H_\pm$ differ only in the parameters appearing in them, a simple algebraic solution scheme for all exactly solvable problems in quantum mechanics can be formulated [5, 6, 7]. We begin this formulation by assuming the form

$$V_+(a, x) = V_-(a_1, x) + L(a_1) \tag{9a}$$

for the relation between the potentials, where a is the set of parameters, a_1 is a function of a ($a_1 = f(a)$) and the remainder $L(a_1)$ is a function independent of x. Relation (9a) implies for the superpotential $W(a, x)$ the equivalent condition

$$W^2(a, x) + W'(a, x) - W^2(a_1, x) + W'(a_1, x) = 2L(a_1). \tag{9b}$$

Equations (9a, b) are the *shape-invariance* conditions for the partner potentials of a supersymmetric Hamiltonian; the self-consistency conditions (9) guarantee a simple algebraic solution for a (supersymmetric) Hamiltonian with a shape-invariant potential. The eigenvalues are, in this case, calculated by constructing formally a series of pairs of Hamiltonians, linked together by conditions (9) for the various potentials. Then the ground states of these Hamiltonians have to be determined; via the (paired) degeneracy of SUSYQM these states give subsequently all excited states of the original Hamiltonian.

Using this scheme, the spectrum of the Hamiltonian

$$\begin{aligned} H &= \tfrac{1}{2}p^2 + V(a, x) \\ &\equiv H_- \end{aligned} \tag{10}$$

is obtained by constructing a series of Hamiltonians H_n, $n = 0, 1, 2, \ldots$, where $H_0 \equiv H_-$ and $H_1 \equiv H_+$. The nth member of the series is the Hamiltonian

$$H_n = \tfrac{1}{2}p^2 + V_-(a_n, x) + \sum_{k=1}^{n} L(a_k) \tag{11a}$$

where a_n is determined by n applications of the function f. In the first step the spectra of H_n and H_{n+1}, where H_{n+1} is defined by

$$\begin{aligned} H_{n+1} &= \tfrac{1}{2}p^2 + V_-(a_{n+1}, x) + \sum_{k=1}^{n+1} L(a_k) \\ &\equiv \tfrac{1}{2}p^2 + V_+(a_n, x) + \sum_{k=1}^{n} L(a_k) \end{aligned} \tag{11b}$$

have to be compared.

Since the Hamiltonians H_n and H_{n+1} in (11) form a supersymmetric pair, their spectra are identical, apart from the ground state E_n^0 of H_n, which is given by

$$E_n^0 = \sum_{k=1}^{n} L(a_k). \tag{12}$$

(Relation (12) can be easily verified using (9a), (11a) and the assumption of vanishing ground-state energy E_-^0 of $H_-(\equiv H_0)$.) On the basis of this argument the degeneracy of the eigenvalues of H_n and H_{n-1} can be compared; an iteration of the pairing argument leads finally to $H_0 = \frac{1}{2}p^2 + V(a, x)$, whose ground-state energy vanishes and whose nth excited level is degenerate with the ground state of the Hamiltonian H_n ($n = 1, 2, \ldots$). Thus the complete spectrum of a Hamiltonian H_-, where

$$\begin{aligned} H_- &= \tfrac{1}{2}p^2 + V(a, x) \\ &= \tfrac{1}{2}p^2 + V_-(a, x) + C(a) \end{aligned} \tag{10'}$$

is given by the formal expression

$$E \equiv \sum_{k=1}^{n} L(a_k) + C(a) \tag{13}$$

(where $C(a)$ denotes an additive constant).

This elementary derivation of the energy levels, based on the shape invariance of the potential $V(a, x)$ and the pairwise degeneracy of supersymmetry, can be applied to *all known exactly solvable problems in quantum mechanics.* (Examples are listed in [1, 6, 7].)

The factorisation method follows a similar scheme, providing a solution for the eigenvalue problem

$$\begin{aligned} H\psi(x, m) &= \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r(x, m)\right)\psi(x, m) \\ &= E\psi(x, m). \end{aligned} \tag{14}$$

The parameter m, (14), is *not* restricted to integer values [4, 5].

Equation (14) is called factorisable, when it can be replaced by *each* of the following two equations:

$$A^-_{m+1}A^+_{m+1}\psi(x, m) = (E - L(m+1))\psi(x, m) \tag{15a}$$

$$A^+_m A^-_m \psi(x, m) = (E - L(m))\psi(x, m). \tag{15b}$$

Here $L(m)$ is an unknown function of the parameter m; the operators $A^\pm_m$ have the form

$$A^\pm_m \equiv \mp\frac{\mathrm{d}}{\mathrm{d}x} + k(x, m) \tag{15c}$$

acting on solutions $\psi(x, m)$ of (14) as

$$A^+_{m+1}\psi(x, m) \simeq \psi(x, m+1) \tag{16a}$$

$$A^-_m\psi(x, m) \simeq \psi(x, m-1). \tag{16b}$$

The unknown function $k(x, m)$ in (15c) can be determined by calculating the products of $A^-_{m+1}A^+_{m+1}$ and $A^+_m A^-_m$ (in (15a, b)) and comparing the results with equation (14). It follows that the eigenvalue problem (14) is factorisable if and only if the function $k(x, m)$, (15c), satisfies the *factorisation condition*

$$k^2(x, m+1) + k'(x, m+1) + k^2(x, m) - k'(x, m) = L(m) - L(m+1). \tag{17}$$

Equation (17) is the basic self-consistency condition of the factorisation method, essentially classifying all factorisable problems [3]. Thus the (classical) factorisation procedure can be interpreted as replacing a given Hamiltonian by a pair of equivalent 'Hamiltonians' with (apart from the ground state) identical spectra, defined by

$$H_+ \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2(x, m+1) + k'(x, m+1) + L(m+1) \tag{18a}$$

$$H_- \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2(x, m) - k'(x, m) + L(m). \tag{18b}$$

Comparing the pair of Hamiltonians in (18) with the supersymmetric pair of (shape-invariant) Hamiltonians in (11) (or the general supersymmetric pair in (3)), we see that the factorisation method is *by definition* based on the replacement of a given Hamiltonian by an equivalent pair of Hamiltonians, constituting a supersymmetric Hamiltonian. This is the key observation, allowing us to summarise the equivalence between both algebraic procedures.

(i) The factorisation condition (17), is completely equivalent to the shape-invariance condition, (9b).

(ii) Since every Friedrichs extension of a minimal Sturm–Liouville operator can be factorised [10], all known exactly solvable problems of quantum mechanics can be transformed to supersymmetric form—the equivalent Hamiltonians in (18) correspond to the supersymmetric pair with shape-invariant potentials given in (11).

(iii) The eigenvalues in problem (14) are determined in complete analogy to the 'supersymmetric' procedure; the ground-state conditions

$$A^{+}_{m_0+1}\psi(x, m_0) = 0 \tag{19a}$$

or

$$A^{-}_{m_0}\psi(x, m_0) = 0 \tag{19b}$$

give immediately the eigenvalue for this (degenerate) level via (15) as

$$E_{m_0} = L(m_0 + 1) \tag{19c}$$

or

$$E_{m_0} = L(m_0) \tag{19d}$$

which corresponds to the procedure discussed for shape-invariant potentials.

(iv) The generation of the eigenfunctions (see below), beginning with a normalised state, preserves normalisation. This can be shown using

$$\begin{aligned}\int (\psi(x, m+1))^2 \,\mathrm{d}x &= \int A^{+}_{m+1}\psi(x, m) A^{+}_{m+1}\psi(x, m)\,\mathrm{d}x \\ &= \int \psi(x, m) A^{-}_{m+1} A^{+}_{m+1}\psi(x, m)\,\mathrm{d}x \\ &= (E - L(m+1)) \int (\psi(x, m))^2 \,\mathrm{d}x.\end{aligned} \tag{20}$$

If the eigenfunction $\psi(x, m+1)$ is normalised, the function $\psi(x, m)$ is also normalised provided the operators $A^{\pm}_{m+1}$ are multiplied with the factor $[E - L(m+1)]^{-1/2}$.

The eigenstates of (14) are generated via the same procedure as in SUSYQM; an iteration loop can be summarised as follows.

We start with the definition of a ground state, assumed to be

$$A^{+}_{m_0+1}\psi(x, m_0) = 0 \tag{21}$$

which gives (cf (19)) the eigenvalue E_{m_0}. All (degenerate) eigensolutions to this eigenvalue are determined by the ansatz:

$$\psi(x, n_0) \equiv \left(\prod_{m=m_0}^{m=n_0+1} A^{-}_{m} \right) \psi(x, m_0) \tag{22}$$

generating a finite sequence of eigenfunctions. In an obvious iteration of this procedure all solutions of (14) can be determined.

The complete equivalence between the factorisation condition (17) and the shape-invariance condition (9b) as well as the equivalence of the corresponding pairs of Hamiltonians ((11) and (18)) is not surprising since it results from a simple extension of a well known theorem in the theory of Riccati equations [8], as we now discuss.

We consider the differential equation

$$\mathrm{d}^2u(x)/\mathrm{d}x^2 - a(x)u(x) = 0 \tag{23a}$$

where the function $a(x)$ is (by assumption) continuous on an interval I of the real line. Let $u(x)$ be a solution of (23*a*) with $u(x)\neq 0$ for a subinterval I_0 of I. Then the logarithmic derivative

$$W(x)\equiv\frac{\mathrm{d}}{\mathrm{d}x}\ln u(x) \tag{23b}$$

is, on I_0, a solution of the Riccati equation

$$W'(x)+W^2(x)=a(x). \tag{23c}$$

(This result can be extended to a linear, homogeneous, coupled first-order system [8].)

The extension is obtained by

(i) introducing parameter-dependent functions in (23);

(ii) subtracting the resulting Riccati equations associated with a pair of equivalent Hamiltonians.

Let us note here that the function $k(x, m)$ in equations (15), (17) and (18) is given by the logarithmic derivative of a solution of (14); this observation allows an easy verification of the statement above [4, 5, 8].

The complete equivalence between the factorisation method and the shape-invariance approach in SUSYQM implies that the range of applications for both procedures necessarily coincides (cf examples listed in Khare and Sukhatme [1] and also [3, 8]); the extension of the factorisation method to continuous eigenvalues is—using the theory of Riccati equations—easy to formulate [5].

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